Improved Gröbner basis computation with applications in cryptography

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- **Improvement 1: Signature-based Gröbner Basis algorithms**
- **Improvement 2: Specialized Gaussian Elimination**
- **Use GB algorithms in algebraic cryptanalysis**

Definition

$G = \{g_1, \ldots, g_r\}$ is a **Gröbner Basis** for $I = \langle f_1, \ldots, f_m \rangle$ if

1. $G \subset I$ and
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Satz (Buchberger's Criterion)

The following are equivalent:

1. *G is a Gröbner Basis for $\langle G \rangle$.*
2. *For all $f, g \in G$ it holds that $\mathrm{spol}(f, g) \xrightarrow{G} 0$, where*

$$\mathrm{spol}(f, g) = \mathrm{lc}(g) \frac{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}{\mathrm{lm}(f)} f - \mathrm{lc}(f) \frac{\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))}{\mathrm{lm}(g)} g.$$

Input: Ideal $I = \langle f_1, \ldots, f_m \rangle$
Output: Gröbner Basis G **for** I

1. $G \leftarrow \emptyset$
2. $G \leftarrow G \cup \{f_i\}$ **for all** $i \in \{1, \ldots, m\}$
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      (ii) **If** $h \xrightarrow{G} r \neq 0 \Rightarrow$ **new information**
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Example

Let $I = \langle g_1, g_2 \rangle \in \mathbb{Q}[x, y, z]$ be given where $\mathbf{g_1} = \mathbf{xy} - \mathbf{z^2}$, $\mathbf{g_2} = \mathbf{y^2} - \mathbf{z^2}$, and let $<$ be the graded reverse lexicographical ordering.

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$$\begin{aligned} \operatorname{spol}(g_2, g_1) &= xg_2 - yg_1 = \mathbf{xy^2} - xz^2 - \mathbf{xy^2} + yz^2 \\ &= -xz^2 + yz^2, \end{aligned}$$

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$\Rightarrow$ **How can we discard such zero reductions in advance?**

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1. Let R^m be generated by $e_1, \ldots, e_m$, $\prec$ a well-ordering on the monomials of R^m, and let $\pi : R^m \to R$ such that

$$\pi(e_i) = f_i \text{ for all } i.$$

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4. **A minimal signature** of p exists due to $\prec$.

Our example – now with signatures and $\prec_{\text{pot}}$

We have already computed the following data:

$$
\begin{aligned}
g_1 &= xy - z^2,\ \operatorname{sig}(g_1) = e_1,\\
g_2 &= y^2 - z^2,\ \operatorname{sig}(g_2) = e_2,\\
g_3 &= \operatorname{spol}(g_2, g_1) = xg_2 - yg_1\\
\Rightarrow \operatorname{sig}(g_3) &= x \operatorname{sig}(g_2) = xe_2.
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$\mathrm{spol}(g_3, g_1) = yg_3 - z^2 g_1$:

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$\Rightarrow$ **We know that $\mathrm{spol}(g_3, g_1)$ will reduce to zero w.r.t. G.**

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The general idea is to check the signatures of the generated s-polynomials.

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Note

We order P by increasing signatures, so we always take the s-polynomial of minimal signature.

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$\mathrm{sig}(h)$ not minimal for h? $\Rightarrow$ Remove h.

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Sketch of proof

1. There exists a syzygy $s \in R^m$ such that $\mathrm{lm}(s) = \mathrm{sig}(h)$.
   $\Rightarrow$ We can represent h with a lower signature.
2. Pairs are handled by increasing signatures.
   $\Rightarrow$ All relations of lower signature are already taken care of.

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Our example with $\prec_{\text{pot}}$ revisited

$\operatorname{sig}(\operatorname{spol}(g_3, g_1)) = xye_2$

$$\left.\begin{array}{l} g_1 = xy - z^2 \\ g_2 = y^2 - z^2 \end{array}\right\} \Rightarrow \operatorname{psyz}(g_2, g_1) = g_1 e_2 - g_2 e_1 = xye_2 + \dots$$

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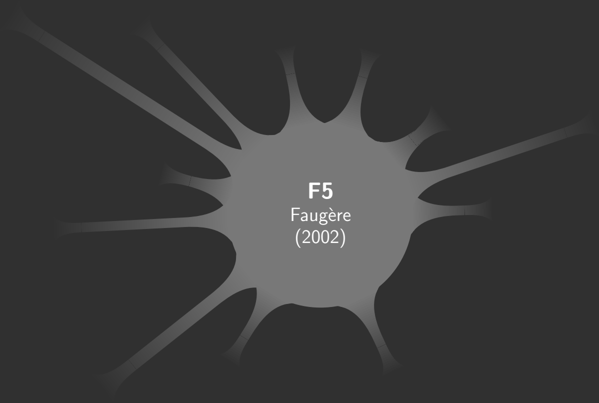
1. $\mathrm{sig}(g - h) \prec \mathrm{sig}(g), \mathrm{sig}(h)$.
2. Pairs are handled by increasing signatures.
   $\Rightarrow$ All necessary computations of lower signature have already taken place.
   $\Rightarrow$ We can represent h by

$$h = g + \text{ elements of lower signature.}$$

$\square$

A good decade on signature-based algorithms



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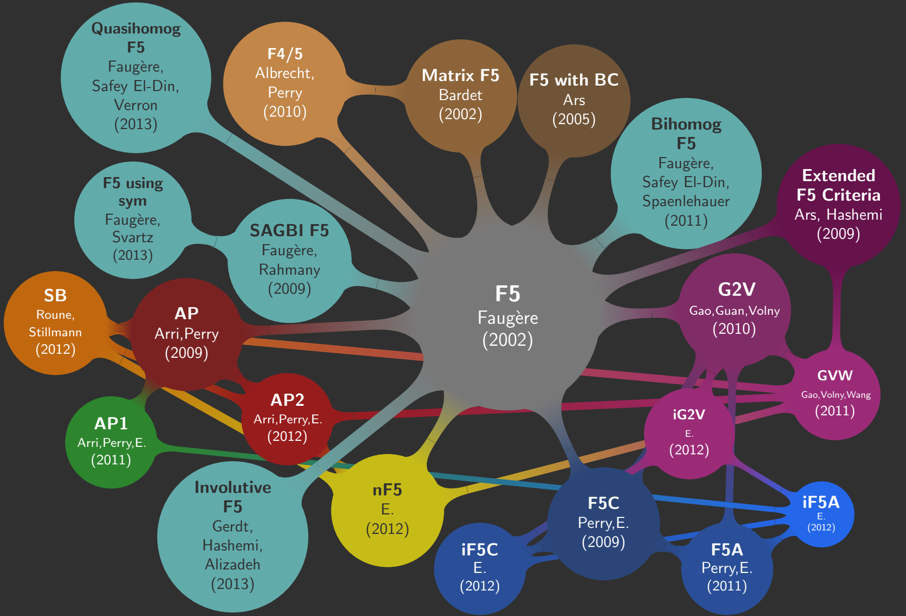

- Improvement 1: Signature-based Gröbner Basis algorithms
- **Improvement 2: Specialized Gaussian Elimination**
- Use GB algorithms in algebraic cryptanalysis

Use **Linear Algebra** for reduction steps in GB computations.

Improve Gaussian Elimination

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$$\begin{matrix} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \\ 0 & 1 & 6 & 0 & 8 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 1 \end{matrix}$$

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Knowledge of underlying GB structure

Improve Gaussian Elimination

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\text{s-polynomial} & \left\{ \begin{array}{ccccccc} 1 & 3 & 0 & 0 & 7 & 1 & 0 \\ 1 & 0 & 4 & 1 & 0 & 0 & 5 \end{array} \right. \\
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Knowledge of underlying GB structure

Idea

Do a static **reordering before** the Gaussian Elimination to achieve a better initial shape. **Reorder afterwards**.

1st step: Sort pivot and non-pivot columns

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2nd step: Sort pivot and non-pivot rows

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\begin{array}{ccc:cccc}
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
0 & 5 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 3 & 1 \\
\hdashline
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
0 & 1 & 8 & 6 & 0 & 0 & 9
\end{array}
$$

Pivot row

Non-Pivot row

3rd step: Reduce lower left part to zero

$$\left.\begin{array}{ccc|cccc}
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
0 & 5 & 0 & 0 & 0 & 2 & 0 \\
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\hline
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
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\end{array}\right.$$

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$$
\left(\begin{array}{ccc:cccc}
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\hdashline
1 & 3 & 7 & 0 & 0 & 1 & 0 \\
0 & 1 & 8 & 6 & 0 & 0 & 9
\end{array}\right)
\longrightarrow
\left(\begin{array}{ccc:cccc}
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
0 & 5 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 3 & 1 \\
\hdashline
0 & 0 & 0 & 7 & 10 & 3 & 10 \\
0 & 0 & 0 & 6 & 0 & 2 & 1
\end{array}\right)
$$

4th step: Reduce lower right part

$$
\begin{array}{ccc|cccc}
1 & 0 & 0 & 4 & 1 & 0 & 5 \\
0 & 5 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 1 & 0 & 0 & 3 & 1 \\
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4th step: Reduce lower right part

$$\left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hline 0 & 0 & 0 & 7 & 10 & 3 & 10 \\ 0 & 0 & 0 & 6 & 0 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 4 & 1 & 0 & 5 \\ 0 & 5 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & 1 \\ \hline 0 & 0 & 0 & 7 & 10 & 3 & 10 \\ 0 & 0 & 0 & 0 & 4 & 1 & 5 \end{array}\right)$$

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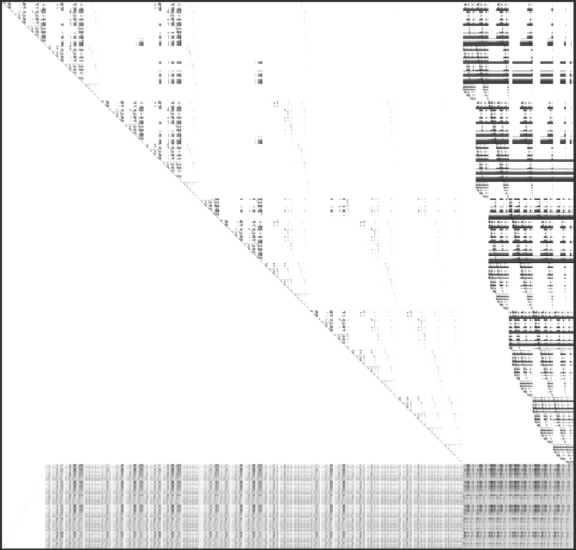
5th step: Remap columns of lower right part

How our matrices look like



Improvements:

- Use knowledge of underlying GB structures
- Parallelization of Linear Algebra
- Divide sparse and dense data as much as possible

Faugère-Lachartre Idea

Improvements:

- Use knowledge of underlying GB structures
- Parallelization of Linear Algebra
- Divide sparse and dense data as much as possible

Recent research:

- Improve parallelization
- Better usage of cache:
  Use small blocks inside matrix per thread
- Use more of the polynomials structure
- Relax idea of signature-based GB algorithms

- Improvement 1: Signature-based Gröbner Basis algorithms

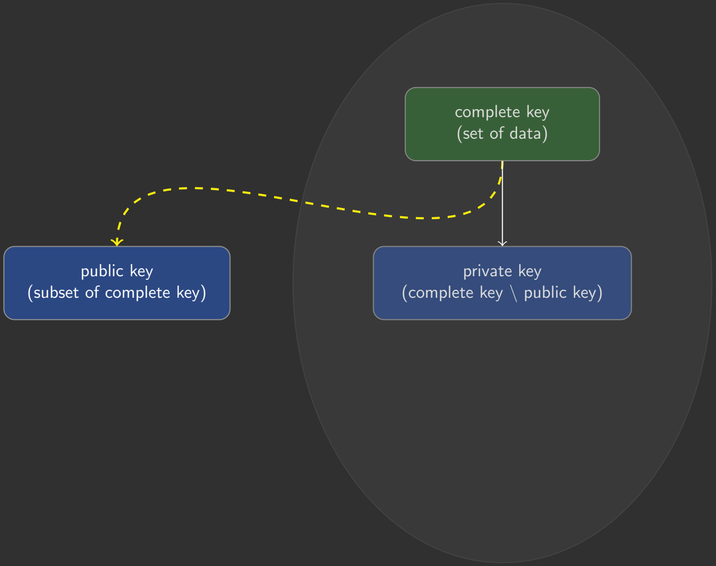
- Improvement 2: Specialized Gaussian Elimination

- **Use GB algorithms in algebraic cryptanalysis**

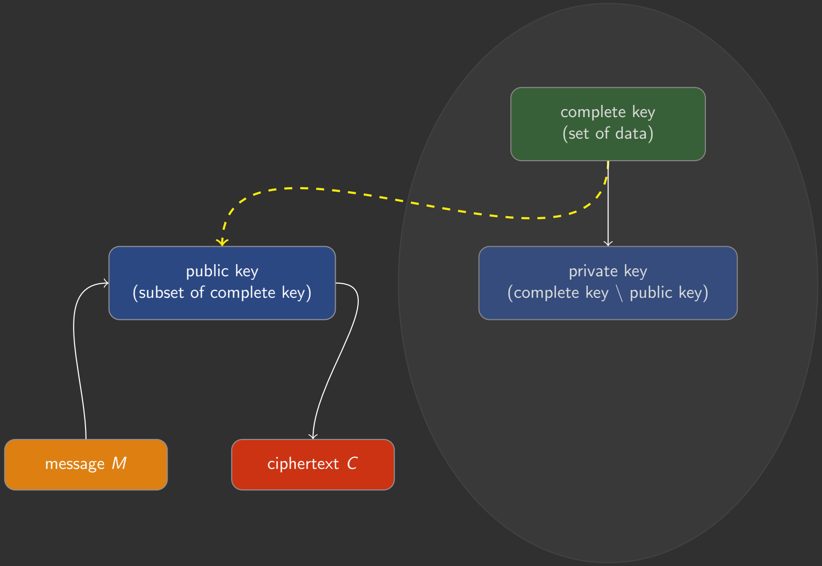
General idea of asymmetric cryptography

complete key
(set of data)

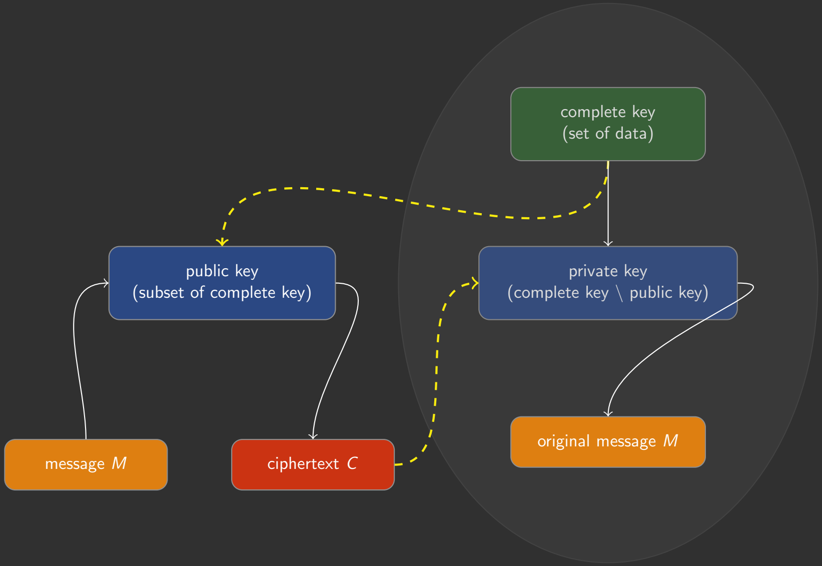
General idea of asymmetric cryptography



General idea of asymmetric cryptography



General idea of asymmetric cryptography



Choice of HFE Polynomial

Choose **private polynomial** p such that

- $p \in F_{q^n}(x)$ (mostly $q = 2$),
- $\deg(p) = d$,
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Common choice:

$$p(x) = \sum_{i,j} \alpha_{i,j} x^{q^{u_{i,j}} + q^{v_{i,j}}} + \sum_k \beta_k x^{q^{w_k}} + \gamma.$$

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system of n quadratic equations in n variables over F_2

Public key: n multivariate polynomials $(p_1, \ldots, p_n)$ over F_q.

HFE Encryption

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- Apply $T \Longrightarrow C = Ty' \in F_q^n$.

HFE Decryption

Simply put: Take C and apply T^{-1}, p^{-1} and S^{-1}.

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How to break the system ?

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How to break the system ?

Solve a system of multivariate quadratic polynomials over F_q:

$$\begin{array}{rcl} p_1(x_1, \ldots, x_n) & = & y_1 \\ \vdots & \vdots & \vdots \\ p_n(x_1, \ldots, x_n) & = & y_n \end{array}$$

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- $d = 96$,
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96 hours of CPU time on an HP workstation with an alpha EV68 processor at 1 GHz and 4 GB RAM
(Whole computation approx. 7.65 GB.)

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